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No. 350

CHANGE OF 180° IN THE DIRECTION OF A UNIFORM CURRENT OF AIR

Contribution by the Aerodynamic Laboratory of the
Warsaw Polytechnic Institute.
Directed by Professor C. Witoszynski

Prepared for publication by J. Bonder

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In the construction of aerodynamic tunnels, it is a very important matter to obtain a uniform current of air in the sections where measurements are to be made. The straight type ordinarily used for attaining a uniform current and generally recommended for use, has great defects. If we desire to avoid these defects, it is well to give the canals of the tunnel such a form that the current, after the change of direction of its asymptotes, approximates a uniform and rectilinear movement. But for this, the condition must be met that at no place does the flow exceed the maximum velocity assumed, equal to the velocity in the straight parts of the canal.

It follows from the above that the problem leads up to the determination of a rational form for the turn, at the change of direction by 180° of a horizontal uniform current.

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* Extract from Przegladu Technicznego, Vol. LXIII, 1925.

Canal Bounded on the Outside by Flat Walls

The plane movement which we are discussing may take place within a plane having the coordinate axes x and y. This being the case, the real part of any function $F(z)$ of the joint change $z = x + iy$ represents the potential of the velocity Φ of a certain plane movement, the condition of ductility being fulfilled, and the imaginary part Ψ represents the actual potential of the movement of the current. The function of the combined change $F(z) = \Phi + i\Psi$ is given the name of combined potential. Each of the lines of the current flowing through may serve as an outline for the walls bounding the canal and the turn of the tunnel.

We get a turn corresponding to the conditions fixed above by determining the combined potential $\Phi + i\Psi$ in the following equation:

$$\sinh\frac{z}{a} = \sinh k \cosh\frac{\Phi + i\Psi}{a\ u} \quad * \qquad (1)$$

in which a is the parameter on a fixed scale, k is an arbitrary parameter, but u is the velocity of the uniform current.

Upon separating the real and the imaginary parts, we get from the above equation the two following ones:

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* C. Witoszynski: Extension and Diffraction of Rays - Lectures in the field of hydrodynamics and aerodynamics. J. Springer, Berlin, 1924.

$$\left.\begin{aligned} \sinh\frac{x}{a}\cos\frac{y}{a} &= \sinh k\cosh\frac{\Phi}{au}\cos\frac{\psi}{au}\ ; \\ \cosh\frac{x}{a}\sin\frac{y}{a} &= \sinh k\sinh\frac{\Phi}{au}\sin\frac{\psi}{au}\ ; \end{aligned}\right\} \qquad (2)$$

In these two equations we establish the potential of the velocity Φ:

$$\left.\begin{aligned} \cosh\frac{\Phi}{au} &= \frac{\sinh\frac{x}{a}\cos\frac{y}{a}}{\sinh k\cos\frac{\psi}{au}}\ ; \\ \sinh\frac{\Phi}{au} &= \frac{\cosh\frac{x}{a}\sin\frac{y}{a}}{\sinh k\sin\frac{\psi}{au}}\ ; \end{aligned}\right\} \qquad (3)$$

by substituting Φ according to these types in the following identity:

$\cos h^2\frac{\Phi}{au} - \sin h^2\frac{\Phi}{au} = 1$, we get the equation of the line of current ψ = constant:

$$\frac{\sinh^2\frac{x}{a}\cos^2\frac{y}{a}}{\sinh^2 k\cos^2\frac{\psi}{au}} - \frac{\cosh^2\frac{x}{a}\sin^2\frac{y}{a}}{\sinh^2 k\sin^2\frac{\psi}{au}} = 1$$

This is transformed according to the two forms of the preceding equations, convenient for discussion:

$$\left.\begin{aligned} \sinh^2\frac{x}{a} &= \frac{\sin^2\frac{y}{a} + \sinh^2 k\sin^2\frac{\psi}{au}}{-\sin^2\frac{y}{a} + \tan^2\frac{\psi}{au}\cos^2\frac{y}{a}}\ ; \\ \sin^2\frac{y}{a} &= \frac{\sinh^2\frac{x}{a} - \sinh^2 k\cos^2\frac{\psi}{au}}{\sinh^2\frac{x}{a} + \cot^2\frac{\psi}{au}\cosh^2\frac{x}{a}}\ ; \end{aligned}\right\} \qquad (4)$$

From type (4) it is evident: 1st., that the change of x to $-x$ does not in itself cause a change of y, or as we have the same arrangement of the lines of current on both sides of the y axis, it is sufficient therefore to turn our attention to only one case, $x > 0$; 2d, that for the value of y differing by $2\pi a$, we will get these same values of x - the arrangement of the lines of current will be repeated near each change of position parallel to the y axis and equal to $2\pi a$.

We will now find the most characteristic form for the lines of current (Fig. 1).

At the x axis ($y = 0$), as is evident from equation (2), either $\Phi = 0$ or $\psi = 0$; that is, at the part of the axis defined by the inequality $x \leqslant ka$, Φ is equal to 0; at the part $x \geqslant ka$, however, ψ is equal to 0. The line of current $\psi = \frac{\pi}{2} au$ is composed, as follows from equation (2), $\sin h \frac{x}{a} \cos \frac{y}{a} = 0$, of three parts:

$$A\,B \quad y = \frac{\pi}{2} a; \qquad B\,C \quad x = 0; \qquad -\frac{\pi}{2} a < y < \frac{\pi}{2} a$$

$$i \quad C\,D \quad y = -\frac{\pi}{2} a.$$

In equation (4) it results from $\sin^2 \frac{y}{a}$ that the lines of current have asymptotes parallel to the x axis and situated at distances from the x axis proportional to the value of the potential of the current ψ. In reality:

$$\underset{x=\infty}{\text{lineal}} \sin^2 \frac{y}{a} = \underset{x=\infty}{\text{lineal}} \frac{1 - \dfrac{\sinh^2 k \cos^2 \dfrac{\psi}{au}}{\sinh^2 \dfrac{x}{a}}}{1 + \cot^2 \dfrac{\psi}{au} \coth^2 \dfrac{x}{a}}$$

$$= \frac{1}{1 + \cot^2 \dfrac{\psi}{au}} = \sin^2 \frac{\psi}{au} ;$$

and therefore, designating by y_a the series of asymptotes for the lines of current ψ = constant, we write

$$y_a = \pm \frac{\psi}{u} . \qquad (5)$$

A diagram of the arrangement of the lines of current is shown in Fig. 1.

Now let us turn to an investigation of the arrangement of velocities. It is easy to get the velocity at arbitrary points in the plane of movement from the form obtained, which results directly as a quality of function of combined change and the limitation of the potentials of velocity and current:

$$v_x - i v_y = \frac{d (\Phi + i\psi)}{d z} . \qquad (6)$$

Because equation (1), determining the combined potential, gives:

$$\Phi + i\Psi = au \operatorname{arc\,cosh}\left(\frac{\sinh \frac{z}{a}}{\sinh k}\right),$$

then

$$v_x - i\,v_y = u\,\frac{\cosh \frac{z}{a}}{\sqrt{\sinh^2 \frac{z}{a} - \sinh^2 k}} \qquad (7)$$

It follows from this that one point at which the velocity becomes infinitely great is the point H $z = ka$. So we see that at infinity the velocity is identical for all lines of current, equal to u. At the vertices of the angles B and C ($x = 0$, $y = \pm\frac{\pi}{2}$) the velocity is equal to zero; $v_x - i\,v_y = 0$, and at the beginning of the arrangement, for this same line of current, O ($z = 0$) is equal to:

$$v_x - i\,v_y = \frac{u}{\sqrt{-\sinh k^2}} = -i\,\frac{u}{\sinh k};$$

or,

$$v_x = 0;\ v_y = \frac{u}{\sinh k};$$

it follows from this that the lengthwise lines of current are the ones the velocity of which passes through the minimum and the maximum.

Because of the application of the condition that on the turn the velocity should nowhere be greater than the velocity in the canal, or than u, we are compelled to determine how the values are reached and where the maximum velocities occur. For this purpose we reproduce the velocities in the form of veloci-

ties of the potential velocity Φ and of current Ψ. We substitute in equation (7) the expression resulting from equation (1) instead of the coordinate function:

$$v_x - i\,v_y = u\,\frac{\sqrt{1+\sinh^2 k\cosh^2\frac{\Phi+i\Psi}{au}}}{\sinh k\sinh\frac{\Phi+i\Psi}{au}};$$

whence we may get the square of the velocity occurring according to the equation: $v^2 = v_x^2 + v_y^2 = (v_x - i\,v_y)(v_x + i\,v_y)$.

$$v^2 = u^2\,\frac{\sqrt{1+\sinh^2 k\left[\cosh^2\frac{\Phi+i\Psi}{au}+\cosh^2\frac{\Phi-i\Psi}{au}\right]+\sinh^4 k\cosh^2\frac{\Phi+i\Psi}{au}\cosh^2\frac{\Phi-i\Psi}{au}}}{\sinh^2 k\sinh\frac{\Phi+i\Psi}{au}\sinh\frac{\Phi-i\Psi}{au}}$$

Because we have:

$$\cosh^2\frac{\Phi+i\Psi}{au}+\cosh^2\frac{\Phi-i\Psi}{au}=1+\cosh\frac{2\Phi}{au}\cos\frac{2\Psi}{au};$$

$$\cosh\frac{\Phi+i\Psi}{au}\cosh\frac{\Phi-i\Psi}{au}=\frac{1}{2}\left(\cosh\frac{2\Phi}{au}+\cos\frac{2\Psi}{au}\right);$$

$$\sinh\frac{\Phi-i\Psi}{au}\sinh\frac{\Phi-i\Psi}{au}=\frac{1}{2}\left(\cosh\frac{2\Phi}{au}-\cos\frac{2\Psi}{au}\right);$$

therefore, it will be:

$$v^2 = u^2\,\frac{\sqrt{4+4\sinh^2 k\left(1+\cosh\frac{2\Phi}{au}\cos\frac{2\Psi}{au}\right)+\sinh^4 k\left(\cosh\frac{2\Phi}{au}+\cos\frac{2\Psi}{au}\right)}}{\sinh^2 k\left(\cosh\frac{2\Phi}{au}-\cos\frac{2\Psi}{au}\right)} \quad (8)$$

In order to find the change of velocity of the lengthwise line of force Ψ = constant, by calculation we derive the square

of the expression (8), or we derive v^4 with regard to Φ. Designating by C the term not influencing the sign determined, we get the simplified form by putting it into effect:

$$C\frac{d(v^4)}{d\Phi} = -\left(\cos^2\frac{2\Psi}{au} + \right.$$

$$\left. + \cos\frac{2\Psi}{au}\cosh\frac{2\Phi}{au} + \frac{2}{\sinh^2 k}\right)\sinh\frac{2\Phi}{au}. \qquad (9)$$

In this equation it is evident that at the intersection of the line of current Ψ = constant with the x axis ($\Phi = 0$), the velocity always passes through an extreme, which is maximum or minimum according to whether the trinomial $T = \cos^2\frac{2\Psi}{au} + \cos\frac{2\Psi}{au} + \frac{2}{\sinh^2 k}$ is positive or negative. If $\sinh^2 k < 8$, then $T > 0$ for each value of Ψ, or then we always have a maximum at the x axis. If, however, $\sinh^2 k \geq 8$, this maximum of velocity at the x axis takes place for those lines of current, the parameter of which fulfills one of the inequalities:

$$\cos\frac{2\Psi}{au} < -\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{2}{\sinh^2 k}}$$

or,

$$\cos\frac{2\Psi}{au} > -\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{2}{\sinh^2 k}}$$

However, it is minimum at the x axis for the lines of current the parameters of which Ψ, are enclosed within the boundaries limited by the inequalities:

$$-\frac{1}{2}-\sqrt{\frac{1}{4}-\frac{2}{\sinh^2 k}}<\cos\frac{2\psi}{au}<$$

$$<-\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{2}{\sinh^2 k}},$$

where then $T < 0$. At the same time, it is easy to convince one's self that in the discussion of the lines of current the other extreme will not occur yet. In reality, in this case, $\cos\frac{2\psi}{au} < 0$, but it follows from the inequality $T < 0$ that

$$-\cos\frac{2\psi}{au} > \cos^2\frac{2\psi}{au}+\frac{2}{\sinh^2 k}$$

and therefore, also

$$-\cos\frac{2\psi}{au}\cosh\frac{2\Phi}{au} > \cos^2\frac{2\psi}{au}+\frac{2}{\sinh^2 k},$$

which, in comparison with formula (9) for the derivation of the velocity, demonstrates that this latter cannot have the second extreme. Because the one extreme first mentioned (equation (9)) may be above the x axis for each line of current, therefore either the velocity does not pass through the maximum or reaches it at the x axis. In this connection, if we wish to convince ourselves that at certain points, bounded by two lines of current $\psi = \psi_1$ and $\psi = \psi_2$, the velocity v that occurs does not exceed a certain value of the above problem, it will be sufficient to confirm the fact that this is not exceeded at the x axis.

On the other hand, the velocity at the x axis, which we designate by V, can easily be obtained from equation (8) by substituting $\Phi = 0$:

$$V = u \frac{\sqrt{1 + \sinh^2 k \cos^2 \frac{\psi}{au}}}{\sinh k \sin \frac{\psi}{au}} \qquad (10)$$

At the point H (Fig. 1), where $\psi = 0$, the velocity $V = \infty$, and subsequently decreases from the form ψ to the value such as was obtained at the beginning of the arrangement O, where $\psi = \frac{\pi}{2} au$, wherefore $V = \frac{u}{\sinh k}$. If therefore we choose k such as to fulfill the inequality $\sin k > 1$, i.e., $k > 0.882$, the middle one of the lines of current will be the one for which the velocity at no point exceeds the velocity u - this line will be on the parameter $\psi > \psi_1$, where ψ_1 is limited by the equation:

$$V = u \frac{\sqrt{1 + \sinh^2 k \cos^2 \frac{\psi_1}{au}}}{\sinh k \sin \frac{\psi_1}{au}} = u$$

whence:

$$\cos \frac{2\psi_1}{au} = - \frac{1}{\sinh^2 k} \qquad (11)$$

From the consideration of the above it follows that the form of an aerodynamic tunnel limited on the outside by flat walls and made according to the conditions specified by us can be described in the following manner: the straight part of the tunnel, that is, the canal, is bounded by the asymptotes

$y_a = \frac{\pi}{2} a$ and $y_o = \frac{\psi_1}{u}$; on the other hand, the turn is bounded by the lines of current with the parameters $\psi = \frac{\pi}{2} au$ and $\psi = \psi_1$; these lines are bound to be intersected by the straight line m - N, perpendicular to the x axis, at such a point that upon comparison of the turn of the canal with the line of current ψ_1, an easy matter in practice, the prolongation is determined by its asymptote $y_a = \frac{\psi_1}{u}$ (Fig. 2).

We will introduce the following designations for the characteristic dimensions of the tunnel (Fig. 2):

1) h - distance from outside wall of canal to axis of tunnel, $h = \frac{\pi}{2} a$;

2) b - clearance of canal, $b = \frac{\pi}{2} a - \frac{\psi_1}{u}$;

3) c - clearance of turn, that is, the distance from the point of intersection of the lines of current bounding the turn with the x axis; the coordinates of these points are designated in equation (2) for $y = 0$, $\Phi = 0$:

$$\sin h \frac{x}{a} = \sin h\, k \cos \frac{\psi}{au} .$$

In regard to saving space, we try to see that the proportions $p = \frac{c}{b}$ and $q = \frac{h}{b}$ are as low as possible, close to unity.

For example, we may write $k = 2$; then in equation (11) $\cos \frac{2\psi_1}{au} = - \frac{1}{\sinh^2 k} = - 0.076$, whence $\psi_1 = 0.823$; therefore the clearance of the canal $b = \frac{\pi}{2} a - \frac{\psi_1}{u} = 0.748\ a$; the remaining clearance of the turn $c = x_1$, where we find x_1 from the

equations $\sinh \frac{x_1}{a} = \sinh k \cos \frac{\Psi_1}{au} = 2.465$, whence $c = x_1 = 1.634$ au; and consequently the relation of the clearance of the turn to the clearance of the canal is found to be: $p = \frac{c}{b} = 2.183$, but on the other hand the proportion $q = \frac{h}{b} = 2.101$. It is shown by these figures that the turn examined requires a great deal of space, and therefore is not suited for large apparatus.

The Canal is Bounded on Either Side by Curved Walls

The cause of the great dimensions of the turn with which we were concerned in the preceding section is the very low velocities in the vicinity of the vertices of the angles B and C. This is expressed analytically by stating that when the lengthwise velocity is studied, certain lines of current first pass through the minimum, to reach the maximum at the x axis. Therefore if we desire to avoid this fault, it is well to see to it that the rate of velocity on one side of the x axis varies in one sense only. To attain this, k becomes equal to $+\infty$. In reality, equation (8) then gives:

$$v^2 = u^2 \frac{\cosh \frac{2\Phi}{au} + \cos \frac{2\Psi}{au}}{\cosh \frac{2\Phi}{au} - \cos \frac{2\Psi}{au}}$$

or

$$v^2 = u^2 + 2u^2 \frac{\cos \frac{2\Psi}{au}}{\cosh \frac{2\Phi}{au} - \cos \frac{2\Psi}{au}} \qquad (13)$$

At the x axis the velocity attains the value $V = u \cot \frac{\psi}{au}$ (13).

From these equations it is evident at once that along the lines of current the parameter of which, $\psi < \frac{\pi au}{4}$, the values of the velocity, at their maximum at the x axis, diminish to u at infinity; however, where $\psi > \frac{\pi au}{4}$, the velocity has its minimum at the x axis, and afterward increases to the value u for $x = \infty$. For the lines of current $\psi = \psi_1 = \frac{\pi au}{4}$ the velocities are identical at all points and equal to u.

For obtaining equations for the new lines of current flowing through, we have to change the arrangement of the coordinates at the point H, in the forms used in the preceding section, or the size of k, and subsequently take $k = \infty$ (Figs. 1 and 3). Equation (1) gives:

$$e^{z/a} = \cosh \frac{\Phi + i\psi}{au} ; \tag{14}$$

whence

$$\left.\begin{aligned} e^{x/a} \cos \frac{y}{a} &= \cosh \frac{\Phi}{au} \cos \frac{\psi}{au} ; \\ e^{x/a} \sin \frac{y}{a} &= \sinh \frac{\Phi}{au} \sin \frac{\psi}{au} ; \end{aligned}\right\} \tag{15}$$

By substituting for Φ in these two equations, we get:

$$e^{\frac{2}{a}x} \frac{\sin^2 \frac{\psi}{au} \cos^2 \frac{\psi}{au}}{\sin^2 \frac{\psi}{au} - \sin^2 \frac{y}{a}} ; \tag{16}$$

or, inversely:

$$\sin^2 \frac{y}{a} = \sin^2 \frac{\psi}{au} \left[1 - e^{-\frac{2x}{a}} \cos^2 \frac{\psi}{au}\right]. \qquad (17)$$

This equation allows the representation of the desired line of current ψ = constant. From its asymptotes, this, like the preceding, will be a straight line: $y_a = \pm \frac{\psi}{u}$ (Fig. 3).

The discussion of the lines of current is very well suited for determining the form of the turn in aerodynamic tunnels. We designate by ψ_1 and ψ_2 the parameters of the lines bounding the turn. The value of ψ_1 is chosen equal to $\frac{\pi}{2}$ au. In equation (13) we obtain the velocities V at the intersection of these lines with the x axis:

$$\left.\begin{aligned} V_1 &= u \cot \frac{\psi_1}{au} = u ; \\ V_2 &= u \cot \frac{\psi_2}{au} = \lambda u. \end{aligned}\right\} \qquad (18)$$

As results from the above equations and from the preceding discussion which has been made, the velocities at points inside the turn will always be contained within the limits $\lambda u < v < u$, where λ may be equal to $\cot \frac{\psi_2}{au} < 1$. From these data we get the equation for the value b of the clearance of the canal:

$$b = \left(\text{arc cot}\, \lambda - \frac{\pi}{4}\right) a. \qquad (19)$$

On the other hand, the clearance of the turn c is equal to $-(x_2 - x_1)$; in equation (16) for $y = 0$, we get:

$$c = x_1 - x_2 = a \ln \frac{\cos \frac{\psi_1}{au}}{\cos \frac{\psi_2}{au}},$$

where, according to equation (18):

$$\cos \frac{\psi_1}{au} = \frac{\sqrt{2}}{2}; \quad \cos \frac{\psi_2}{au} = \frac{\lambda}{\sqrt{1 + \lambda^2}};$$

wherefore

$$c = a \ln \frac{\sqrt{2(1 + \lambda^2)}}{2}. \qquad (20)$$

For example, the following dimensions for the turn are obtained by adhering to the condition that the velocity on the turn should nowhere be less than half the velocity u in the canal ($\lambda = \frac{1}{2}$) (Fig. 4):

$$h = a \operatorname{arc\,cot} \tfrac{1}{2} = 1.109\ a;$$

$$b = \left(\operatorname{arc\,cot} \tfrac{1}{2} - \tfrac{\pi}{4}\right) a = 0.323\ a;$$

$$c = a \ln \frac{\sqrt{2(1 + \lambda^2)}}{2\lambda} = 0.458\ a;$$

whence

$$p = \frac{c}{b} = 1.42; \qquad q = \frac{h}{b} = 3.43.$$

As is evident from the above example, such a tunnel has this disadvantage, that in comparison to the clearance of the canal its branches are at a great distance from each other, or that the proportion of unused space is large.

This defect may be avoided by having the tunnel composed

of several simple canals, in the manner shown in Fig. 5. The method of procedure is the following: The limit lines of current in the canal ψ_1 and ψ_2 may be marked by a scale $a = a_1$. We now change the scale to $a = a_2$, choosing it in such a way that the position of the asymptote $\frac{2\psi_1}{u}$ of the lines ψ_1 (scale a_2) is equal to the position of the asymptote of the lines ψ_2 (scale a_1). Consequently, at infinity these lines of current ψ_1 (a_2) and ψ_2 (a_1) will be near each other. Then if we shift lengthwise the axis of the canal common to the scale a_2 and the scale a_1, it will be easy to find the position near which these lines of current will be visibly separated from each by gradually proportionate intervals. By putting this form into execution and proceeding further in a similar way we get a tunnel very suitable for current flowing through in a semilunar direction. In comparison with the preceding tunnels not having such form, this latter is interesting in that it requires less space crosswise. And in addition it has the further advantage, thanks to the semilunar character of its form, of preventing the shearing or cutting action of the air on the wall of the tunnel.

General Case of a Canal with Curved Walls

We will now take up a general discussion of the combined potential (14); in particular, we shall adhere to the nomenclature previously accepted; we shall define the combined poten-

tial by the equation:

$$\cosh \frac{\Phi + i\psi}{au} = \frac{2\, e^{z/a} - m - 1}{1 - m} \qquad (21)$$

By $m = -1$ we get the potential (14) sought for. As we can convince ourselves, from this potential comes the arrangement of the lines of current, having a whole series of interesting properties.

We will limit ourselves to finding the flow for $|m| < 1$, when, if $|m| > 1$, it will suffice to take the parameter $m_1 = \frac{1}{m}$ in equation (21) instead of the parameter m, and we get

$$\cosh \frac{\Phi + i\psi}{au} = \frac{2\, e^{z/a} - \frac{1}{m} - 1}{1 - \frac{1}{m}},$$

or,

$$\cosh \frac{\Phi + i(\psi + \pi\, au)}{au} = \frac{2\, e^{\frac{z}{a} + \ln m} - 1 - m}{1 - m},$$

that is, this flow differs from the preceding flow with parameter m only in that the arrangement of coordinates is shifted by $z = a \ln m$, and the values of ψ for the corresponding lines of current are diminished by $\pi\, au$.

Bearing in mind then that $|m| < 1$, we determine the arrangement of velocities from equation (21):

$$v_x - i\, v_y = u \frac{e^{z/a}}{\sqrt{(e^{z/a} - m)\,(e^{z/a} - 1)}}, \qquad (22)$$

from which it follows that the velocity becomes infinitely large at the points $\frac{z}{a} = 2 k \pi i$, or $\frac{z}{a} = \ln m + 2 k \pi i$, where k is an arbitrary whole member. It is well to distinguish two cases here: $0 < m < 1$ and $-1 < m < 0$. In the first case the poles (that is, the points at which the velocity is infinitely great) lie on the straight lines parallel to the x axis and $2 \pi a$ apart from each other; in the second one the poles are arranged in checkerboard fashion and the series of poles $\frac{z}{a} = 2 k \pi i$ remains in its place, while the other series $\frac{z}{a} = \ln m + 2 k \pi i = \ln (-m) + \pi i + 2 k \pi i$ is moved toward the top or the bottom by πa.

We will first take up the case in which $0 < m < 1$.

Because the arrangement of the lines of current is repeated upon movement in the direction of the y axis by $2 \pi a$, it is sufficient to consider y only within the limits $- \pi a < y < \pi a$.

In equation (21), by equating the different real and imaginary parts, we get:

$$\left.\begin{aligned} \cosh \frac{\Phi}{au} \cos \frac{\psi}{au} &= \frac{2e^{x/a} \cos \frac{y}{a} - m - 1}{1 - m}, \\ \sinh \frac{\Phi}{au} \sin \frac{\psi}{au} &= \frac{2e^{x/a} \sin \frac{y}{a}}{1 - m} \end{aligned}\right\} \qquad (23)$$

and therefore by substituting for Φ and ψ we derive in the following the equation for the lines of current ψ = constant:

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$$4\left(\cos^2\frac{y}{a} - \cos^2\frac{\psi}{au}\right)e^{\frac{2x}{a}} +$$

$$- 4\,(m + 1)\cos\frac{y}{a}\left(1 - \cos^2\frac{\psi}{au}\right)e^{\frac{x}{a}} +$$

$$+ \left[(1 + m)^2 - (1 - m)^2\cos^2\frac{\psi}{au}\right]\left(1 - \cos^2\frac{\psi}{au}\right) = 0. \qquad (24)$$

From equation (23) it is evident that the line of current $\psi = 0$ is created on the positive side of the x axis ($y = 0$; $x > 0$). The line of current $\psi = \pi\, au$ is composed of part of the x-axis, namely, $y = 0$, $x < a \ln m$ and of two straight lines parallel to the axis; $y = \pm\pi a$ (Fig. 6). At the intersection of the x axis, for $a \ln m < x < 0$ we have $\Phi = 0$; in the remaining points, on the other hand, the potential Φ is defined by the equation:

$$\sinh\frac{\Phi}{au} = \frac{2e^{x/a}\sin\frac{y}{a}}{(1 - m)\sin\frac{\psi}{au}} \qquad (25)$$

In equation (22) we read at once that

where $x = +\infty$ we have $v_x - i\,v_y = \pm u$,

and where $x = -\infty$ we have $v_x - i\,v_y = 0$, (26)

or at $+\infty$ we have a uniform current of air and at $-\infty$ this current turns back. If we desire to limit the character of the lines of current, we raise both sides of equation (22) to a square:

$$v_x^2 - v_y^2 - 2 i v_x v_y = u^2 \frac{e^{\frac{2z}{a}}}{(e^{z/a} - m)(e^{z/a} - 1)},$$

whence we get:

$$v_x v_y = \frac{u^2 e^{\frac{2x}{a}} \left[(1 + m) e^{\frac{x}{a}} - 2m \cos \frac{y}{a}\right] \sin \frac{y}{a}}{2\left[\left(e^{x/a} - m \cos \frac{y}{a}\right)^2 + m^2 \sin^2 \frac{y}{a}\right]\left[\left(e^{x/a} - \cos \frac{y}{a}\right)^2 + \sin^2 \frac{y}{a}\right]} . \quad (27)$$

As is evident from this equation, the denominator is always positive, wherefore in reality the sign of the product of the velocities $v_x v_y$ depends only on the sign of the expression:

$$L = \left[(1 + m) e^{x/a} - 2m \cos \frac{y}{a}\right] \sin \frac{y}{a} . \quad (28)$$

It is evident in reality that the side on which the curves are tangent to the line of current is limited entirely by the sign of the product $v_x v_y$, independent of the signs of the separate factors, the knowledge of which is necessary just at the time when we take up the determination of the expression for the flow, that is, the determination of whether the flow takes place from A to B or from B to A. This, however, remains arbitrary and properly depends on what sign we place before the roots in equation (22). Therefore it is sufficient for determining the character of the arrangement of the lines of current to find out the sign of the product $v_x v_y$ and this only for $y > 0$, when this arrangement is symmetrical with re-

gard to the x axis. From this assumption it follows in equation (27) that for $x > 0$ we have without exception $v_x v_y > 0$; however, for $x < 0$, that is $e^{x/a} < 1$, it is well to distinguish two cases (Fig. 6):

1) $\cos \frac{y}{a} < \frac{(1 + m)\, e^{x/a}}{2\, m}$, then $v_x v_y > 0$;

2) $\cos \frac{y}{a} > \frac{(1 + m)\, e^{x/a}}{2\, m}$, then $v_x v_y < 0$.

And, consequently, when $\cos \frac{y}{a} = \frac{(1 + m)\, e^{x/a}}{2\, m}$, we then have $v_x v_y = 0$, and properly, as we can be convinced by a look at Fig. 6, $v_x = 0$. Then we say that the equation

$$\cos \frac{y}{a} = \frac{1 + m}{2\, m}\, e^{x/a} \tag{29}$$

expresses the geometrical position of the points for which $v_x = 0$ (in Fig. 6, curve MNK).

This curve intersects the x axis at the point N of the backward flow

$$\frac{x}{a} = \ln \frac{2\, m}{1 + m} \tag{30}$$

and has the asymptotes $y = \pm \frac{\pi}{2}\, a$.

We will also add the arrangements of velocities on the x axis at the interval $a \ln m < x < 0$. In equation (22), for $y = 0$ and x enclosed within the above interval, we get:

$$v_x = 0; \quad v_y = V = \frac{u\, e^{\frac{x}{a}}}{\sqrt{(e^{\frac{x}{a}} - m)\,(1 - e^{\frac{x}{a}})}} \tag{31}$$

As we can easily convince ourselves by a simple differential, this velocity passes through the minimum, where:

$$e^{\frac{x}{a}} = \frac{2\,m}{1+m},$$

and therefore the minimum of the velocity at the x axis takes place at the point N, at which it intersects the line MNK (Fig. 6). The line of current ψ_g which passes through this point we call the limit line. The value of its parameter ψ_g is determined from equation (23); by applying the substitutions

$$y = 0; \quad \Phi = 0; \quad e^{\frac{x}{a}} = \frac{2\,m}{1+m};$$

we get

$$\cos\frac{\psi_g}{au} = -\frac{1-m}{1+m}; \tag{32}$$

because

$$0 < m < 1, \quad \text{therefore,} \quad \frac{\pi}{2} < \frac{\psi_g}{au} < \pi.$$

The minimum velocity at the x axis has the velocity (equation 31):

$$V_{min} = \frac{2\,u}{1-m}\sqrt{m}. \tag{33}$$

If we wish to have the interval at the x axis at which the controlling velocity decreases from the velocity u of the uniform current, such as we have it, to the factor for $x = +\infty$, it is well to make $V_{min} < u$, that is, $\frac{2\sqrt{m}}{1-m} < 1$, whence:

$$0 < m < 3 - 2\sqrt{2} \approx 0.1716 \tag{34}$$

If this condition is fulfilled, there are for the arrangement of the lines of current examined, two with parameters ψ_1 and ψ_2, at the intersection of which with the x axis the velocity $V = u$. For the purpose of determining the values of these parameters, we substitute in equations (31) and (23) for the variable x, which on the assumption that $y = 0$, $\Phi = 0$, leads to the following equation for the velocity V at the x axis:

$$V = u \frac{(1 - m) \cos \frac{\psi}{au} + 1 + m}{(1 - m) \sin \frac{\psi}{au}} \tag{35}$$

Taking $V = u$, we get the equation for the determination of the parameters ψ_1 and ψ_2:

$$\sin\left(\frac{\psi_{1,2}}{au} - \frac{\pi}{4} \right) = \frac{1 + m}{1 - m} \frac{\sqrt{2}}{2} \tag{36}$$

$$\frac{\sqrt{2}}{2} < \frac{1 + m}{1 - m} \frac{\sqrt{2}}{2} < 1,$$

whence

$$\frac{\pi}{4} < \frac{\psi_{1,2}}{au} - \frac{\pi}{4} < \frac{3}{4}\pi \quad \text{also:} \quad \frac{\pi}{2} < \frac{\psi_{1,2}}{au} < \pi .$$

The values of the parameters which are satisfied in equation (36) are collected in the form:

$$\psi_1 + \psi_2 = \frac{3}{2} \pi \, au \tag{37}$$

We take (Fig. 7) the turn for the aerodynamic tunnel from

these two lines of current, the parameters ψ_1 and ψ_2 being supplemented in such a way that the relation q between the distance h from the outside wall of the canal to the axis of the tunnel and between the clearance of the canal b should, for the sake of facility of comparison, be the same as in the example, a diagram of which is given in Fig. 4. And therefore, $q = \frac{h}{b} = \frac{\psi_2}{\psi_2 - \psi_1} = 3.43$; whence in combination with equation (37) we get

$$\psi_1 = 0.622\ \pi \text{ au}, \qquad \psi_2 = 0.878\ \pi \text{ au},$$

which allows the clearance of the canal to be fixed:

$$b = \frac{\psi_2 - \psi_1}{u} = 0.804\ a,$$

and the value of the parameter $m = 0.131$ to be fixed by equation (36). These data already limit the whole system of the lines of current. So then by reenforcing equations (23), we define the clearance of the turn c:

$$c = -(x_2 - x_1) = 0.905\ a.$$

It is well to remark here, nevertheless, that because the line $\psi = \psi_2$ has convexity in the region of the point M (Fig. 7), therefore, if we are going to make a comparison of the two turns in regard to space occupied, the dimension c is not measured, but c', by which we designate the breadth of the turn: $c' = -(x_M - x_1)$. Here x_M indicates the turning downward at the point M, the coordinates of which are elements

of the arrangement of types (24) and (29), in which $\psi = \psi_2 = 0.878\ \pi$ au. By transposing the calculations we get: $c' = 1.086\ a$, whence the relation of the breadth of the turn to the clearance of the canal $p' = \frac{c'}{b} = 1.35$, and therefore, is somewhat better than in the example with which we are making a comparison (there we get $p = 1.42$).

We will now proceed to a discussion of the case: $-1 < m < 0$.

The forms of the preceding equations still remain valid; only in connection with the fact that m is now negative do we get other results. And therefore we read in equation (32) that one pole is, as in the preceding, at the point $z = 0$; however, the other, defined by the equation $e^{\frac{z}{a}} = m$, has the coordinates: $x = a \ln(-m)$; $y = \pi a$. Equation (23) gives the position of the principal lines of current ψ and likewise of the potential of the velocity Φ:

$\psi = 0$ - the positive side of the x axis, or $y = 0$; $x > 0$.

$\psi = \pi$ au - the straight part parallel to the x axis, that is, $y = \pi a$; $x > a \ln(-m)$;

$\Phi = 0$ - the remaining parts other than straight, that is, $y = 0$, $x < 0$ or $y = \pi a$, $x < a \ln(-m)$.

Passing to the examination of the arrangement of velocities, we have the same as those for m, with the addition that the sign of the product of the combined velocities $v_x v_y$, determining on which side of the slope the lines of current are, is identical with the same of expression (28):

$$L = \left[(1 + m)\ e^{\frac{x}{a}} - 2\ m \cos \frac{y}{a} \right] \sin \frac{y}{a} .$$

In connection with the fact that $-1 < m < 0$,

$$v_x\, v_y > 0, \quad \text{as far as} \quad \cos \frac{y}{a} > \frac{1 + m}{2\ m}\ e^{\frac{x}{a}} ;$$

however, $v_x\, v_y < 0$, when $\cos \frac{y}{a} < \frac{1 + m}{2\ m}\ e^{\frac{x}{a}}$.

From these relations result the diagram of the arrangement of lines of current given in Fig. 8. The curve MNK is the geometrical position of the points at which $v_y = 0$. The equation for it is the following:

$$\cos \frac{y}{a} = \frac{1 + m}{2\ m}\ e^{\frac{x}{a}} \tag{38}$$

This curve intersects the straight line $y = \pi\ a$ at the point N, determined from the equation: $e^{\frac{x}{a}} = \frac{-2\ m}{1 + m}$; whence $x = a \ln (-m) + a \ln \frac{2}{1 + m}$; its asymptotes are straight lines: $y_a = \frac{\pi}{2}\ a$ and $y_a = \frac{3\pi}{2}\ a$.

As is evident from Fig. 8, the lines of current are divided into two groups, separated from each other by the boundary line ψ_g, which runs from $-\infty$ to $+\infty$. And, therefore, in order to get the equation for its parameter ψ_g, it is well in equation (23)

$$\cosh \frac{\Phi}{au} \cos \frac{\psi}{au} = \frac{2e^{\frac{x}{a}} \cos \frac{y}{a} + m - 1}{1 - m}$$

to take $\Phi = 0$, $\frac{x}{a} = -\infty$, $y = 0$; at once we find that:

$$\cos \frac{\psi_g}{au} = -\frac{1 + m}{1 - m} < 0.$$

We have therefore $\frac{\pi}{2} < \frac{\psi_g}{au} < \pi$. When we take $m = -1$, we get $\psi_g = \frac{\pi}{2} au$ and we reach the flow previously examined; where $m = 0$, on the contrary, ψ_g will be equal to $\pi\, au$ and we shall have the factors which are evident in equation (21), only on another scale.

By substituting equation (39) for equation (24), we get the equation for the boundary line $\psi = \psi_g$:

$$e^{x/a} = \frac{-4\, m\, (m+1)\, \cos \frac{y}{a}}{(m-1)^2\, \cos^2 \frac{y}{a} - (m+1)^2} \qquad (40)$$

from which it is evident that the boundary line possesses two asymptotes (Fig. 8):

1) $x = +\infty$; $\cos \frac{y}{a} = -\frac{1+m}{1-m} = \cos \frac{\psi_g}{au}$

whence $y_a = \frac{\psi_g}{u} > \frac{\pi}{2} a$;

2) $x = -\infty$; $\cos \frac{y}{a} = 0$, and therefore $y_a = \frac{\pi}{2} a$.

We will also remark here that the lines of current situated above the boundary line attain greater breadth than those contained within their asymptotes (Fig. 8). We designate by the extreme limit of the point of the line of current on the straight line $y = \pi a$; this limit is greater than the limit for the corresponding asymptotes on the straight line $y = \pi a$. To obtain the coordinates of the point spoken of, which lies at

the intersection of the line of current considered with the curve MNK, it is well to work out the system of equations (24) and (38).

There still remains for discussion the variation of the velocity of the flow near the x axis and the straight line $y = \pi a$. It follows from equation (22) that at the x axis, for $y = 0$, $x < 0$, the velocity V will be:

$$V = v_y = \frac{u\, e^{\frac{x}{a}}}{\sqrt{(e^{\frac{x}{a}} - m)\,(1 - e^{\frac{x}{a}})}}$$

Thus it is easy to convince one's self that this velocity diminishes from an infinitely great value at the point $x = 0$, $y = 0$, to zero for $x = -\infty$.

Analogically, we find the velocity V on the straight line $y = \pi a$, where $z = x + i\pi a$, from the factor $x < a \ln(-m)$; by executing a simple transformation, we reach the equation:

$$V = v_y = -\frac{u\, e^{\frac{x}{a}}}{\sqrt{(1 + e^{\frac{x}{a}})\,(-m - e^{\frac{x}{a}})}}$$

And here the velocity decreases from an infinitely great value at the point $z = a \ln(-m) + i\pi a$ to zero. Both of these formulas may be combined into one, taking into consideration the fact that according to equation (23) for $\Phi = 0$ and $y = 0$ we have:

$$e^{\frac{x}{a}} = \frac{(1 - m) \cos \frac{\psi}{au} + 1 + m}{2}$$

but on the other hand, for $\Phi = 0$ and $y = \pi a$ there will be:

$$e^{\frac{x}{a}} = - \frac{(1 - m) \cos \frac{\psi}{au} + 1 + m}{2}$$

In either case therefore we will get:

$$V = u \frac{(1 - m) \cos \frac{\psi}{au} + 1 + m}{(1 - m) \sin \frac{\psi}{au}} \qquad (41)$$

From the above equation we can fix the parameter ψ_1 or ψ_2 of the line of current limited by the condition that the velocity at its point of intersection with the x axis or the straight line $y = \pi a$ must be equal to the velocity u existing at the points on the segment $x = + \infty$. It is necessary only to take into consideration in Fig. 8 the course of the flow to be found from the first case $V = u$ and from the second case $V = - u$. By executing an easy transformation we get:

$$\sin\left(\frac{\psi_1}{au} - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \frac{1 + m}{1 - m}; \qquad (42)$$

$$\cos\left(\frac{\psi_2}{au} - \frac{\pi}{4}\right) = - \frac{\sqrt{2}}{2} \frac{1 + m}{1 - m}. \qquad (43)$$

Bearing in mind that $-1 < m < 0$, we will get the following limits for the parameters ψ_1 and $i\psi_2$:

$$\frac{\pi}{4} < \frac{\psi_1}{au} < \frac{\pi}{2}; \quad \frac{3}{4}\pi < \frac{\psi_2}{au} < \pi.$$

These parameters are combined into a simple expression:

$$\frac{\psi_2}{au} - \frac{\psi_1}{au} = \frac{\pi}{2} . \qquad (44)$$

The formulas worked out already will enable us to solve the problem of finding the dimensions of tunnels, limited by the two lines of current flowing through which have been examined. It is evident that both of these lines must belong to the same group of curves, that is, they are situated either under the boundary curve ψ_g or else above it.

For the sake of a better understanding, we will give several numerical examples.

We make the first tunnel with curves situated below the boundary line. Thus a sketch of one wall may serve for the line of current ψ_1; from the sketch of the other one, the line of current ψ_λ may be defined, on condition that at its intersection with the x axis the velocity is equal to λu. If we desire to have material for comparison with the different examples of flow through previously given, we apply the condition in the first place that the factor $q = \frac{h}{b}$ should possess the same value as it had in the preceding examples, that is, $q = 3.43$, whence it follows that

$$\psi_\lambda = 1.41\ \psi_1$$

Example I: $m = -0.5$. From equation (42), $\psi_1 = 1.02$ au, and consequently, $\psi_\lambda = 1.44$ au, whence the clearance of the

canal, $b = 0.42\ a$; on the other hand, the clearance of the turn c is found to be equal to $-(x_\lambda - x_1)$; by determining the quantities x_λ and x_1 from equation (23), we then get $c = 0.61\ a$, wherefore $p = \frac{c}{b} = 1.46$; the arrangement of the velocities of the parts is shown by the expression: $\lambda = 0.46$ from equation (41)).

Example II: $m = -0.2$. Proceeding in a similar way to the preceding, we get: $\psi_1 = 1.28\ au$; $\psi_\lambda = 1.80\ au$; $b = 0.52\ a$; $c = 0.82\ a$; $p = 1.57$; $\lambda = 0.45$.

For $m = 0$ or $m = -1$ we should get such dimensions as in the example for the potential (14).

And now for these same values of m we form a tunnel with lines of current situated below the boundary line; then in this case we shall be bound to fulfill the condition:

$$\pi\, a - \frac{\psi_\lambda}{u} = 1.41 \left(\pi\, a - \frac{\psi_2}{u} \right).$$

Example III: $m = -0.5$; $\pi - \frac{\psi_2}{au} = 0.55$; $\pi - \frac{\psi_\lambda}{au} = 0.77$; $b = 0.22\ a$; $c = 0.30\ a$; $p = 1.33$; $\lambda = -0.56$. Moreover, we should add here the measurement $l_2 = 0.62\ a$ for the curve $\psi = \psi_2$, and the corresponding one $l_\lambda = 0.95\ a$ for $\psi = \psi_\lambda$.

Example IV: $m = -0.2$; $\pi - \frac{\psi_2}{au} = 0.29$; $\pi - \frac{\psi_\lambda}{au} = 0.41$; $b = 0.12\ a$; $c = 0.15\ a$; $p = 1.24$; $\lambda = -0.63$; $l_2 = 0.39\ a$; $l_\lambda = 0.57\ a$.

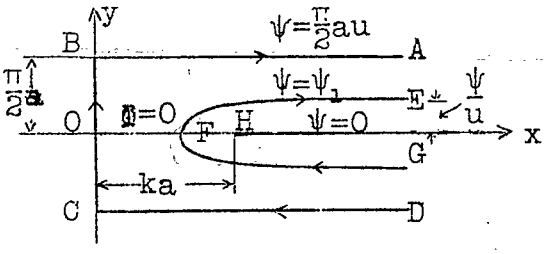

Fig.1

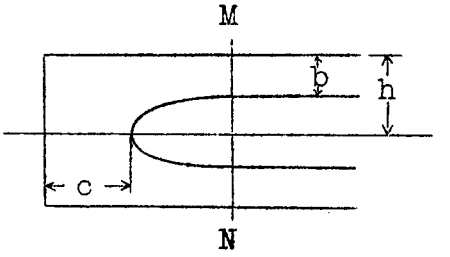

Fig.2

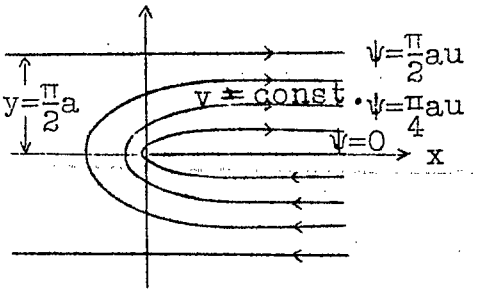

Fig.3

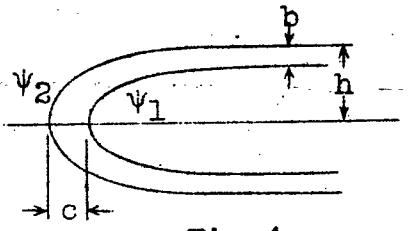

Fig.4

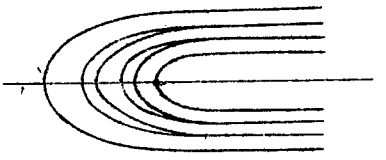

Fig.5

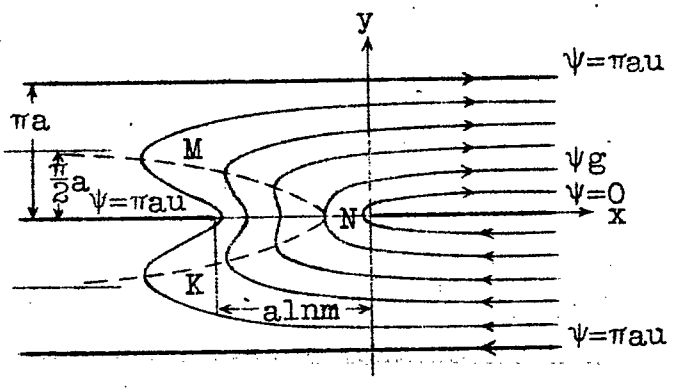

Fig.6

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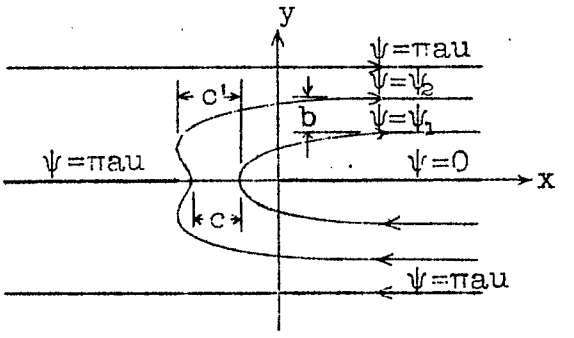
y
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b
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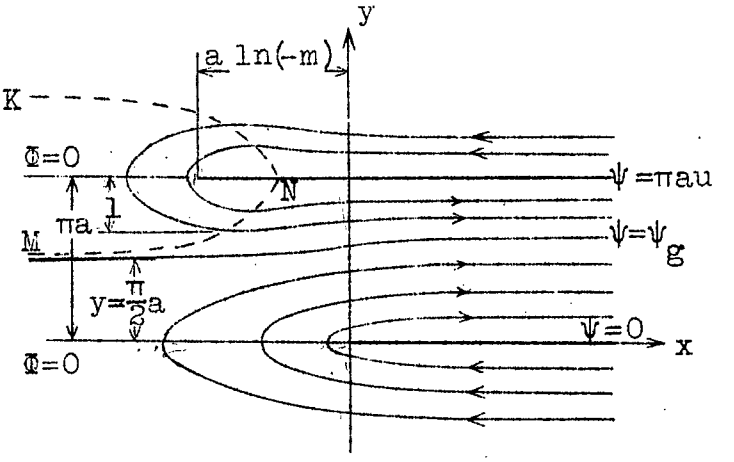
ψ=πau

Fig.7


y
a ln(-m)
K
Φ=0
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ψ=ψg
y=π/2 a
ψ=0
x
Φ=0

Fig.8

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